Other classes of Distributions.

Denote by $\mathcal{E}' \subseteq \mathcal{D}'$ the space of distr. of compact support.

Recall. • $(\operatorname{Supp} F)^c = \bigcup \{ U_\alpha : U_\alpha \text{ open}, F|_{C_c^\infty(U_\alpha)} = 0 \}$. Thus, if $\varphi \in C_c^\infty(K)$, $K \subseteq (\operatorname{Supp} F)^c$ then $\langle F, \varphi \rangle = 0$.

• $\mathcal{E}'$ played a role in the pf that C_c^∞ is dense in $\mathcal{D}'$. We argued that if $\varphi \in C_c^\infty$, $\int \varphi = 1$, $F \in \mathcal{E}'$, then $F * \varphi_t \in C_c^\infty$, $F * \varphi_t \to F$ in $\mathcal{D}'$. Then, we argued that $\mathcal{E}'$ is dense in $\mathcal{D}'$.

$\mathcal{E}'$ can itself be realized as a space of linear, cont. functionals.

- Endow C^∞ w/ the Fréchet topology s.t. $\varphi_j \to \varphi$ in $C^\infty \iff \partial^\alpha \varphi_j \to \partial^\alpha \varphi$ unif. on every compact and for any $\alpha \in \mathbb{Z}_+^n$.

<u>Thm 1.</u> $\mathcal{E}'$ can be identified w/ dual of $\mathcal{E} = C^\infty$.

Next, the space of <u>tempered distributions</u>, $\mathcal{S}'$, is the dual of the Fréchet space $\mathcal{S}$.

Since we have, as spaces,

$$C_c^\infty \subseteq \mathcal{S} \subseteq C^\infty, \quad \left(\begin{array}{l}\text{w/ increasing collections}\\ \text{of conv. sequences.}\end{array}\right)$$

we have $\mathcal{E}' \subseteq \mathcal{S}' \subseteq \mathcal{D}'$. The usual operations work in each space of distributions. One comment regarding multiplication in $\mathcal{S}'$.

The space of C^∞ fcns w/ <u>slow growth</u> (or tempered growth) comprises those $\varphi \in C^\infty$ s.t. $\exists\, C_\alpha, N_\alpha$ s.t.

$$|(\partial^\alpha \varphi)(x)| \leq C_\alpha (1+|x|)^{N_\alpha}.$$

The multiplication operator $M_\varphi F = \varphi F$, for $F \in \mathcal{S}'$, must be restricted to $\varphi \in C^\infty$ w/ slow growth.

Fourier Transform on $\mathcal{S}'$

Now, we may define Fourier transform $\mathcal{F}$ on $\mathcal{S}'$ by duality. Recall $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is an $\underset{\text{(lin. + cont.)}}{\text{isomorphism}}$ $f \to \hat{f}$ w/ inverse $f \to f^{\vee}$. For $F \in \mathcal{S}'$

$$\begin{cases} \langle \hat{F}, \varphi \rangle = \langle F, \hat{\varphi} \rangle, \\ \langle F^{\vee}, \varphi \rangle = \langle F, \varphi^{\vee} \rangle, \end{cases} \quad \forall \varphi \in \mathcal{S}.$$

Note that this definition is consistent w/ $f \to \hat{f}$ for $f \in L^1$ by Plancherel's formula

$$\int \hat{f} g = \int f \hat{g}, \quad f, g \in L^1$$

<u>Thm2</u> $F \to \hat{F}$ is an isomorphism $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ w/ inverse $F \to F^\vee$.

Since $\mathcal{E}' \subseteq \mathcal{S}'$, we also have $\mathcal{F}$ on $\mathcal{E}'$. Moreover, since $E_\xi(x) = e^{2\pi i \xi \cdot x} \in C^\infty$, we can also define $f(\xi) = \langle F, E_{-\xi} \rangle$ for $F \in \mathcal{E}'$.

<u>Prop1.</u> Let $F \in \mathcal{E}'$ and define $f = \langle F, E_{-\xi} \rangle$. Then $f \in C^\infty$ w/ slow growth and $\langle \hat{F}, \varphi \rangle = \int f \varphi$.

Sobolev spaces.

Recall that smoothness of a function is detectable by rate of decrease of its Fourier transform,

- $(\partial^\alpha f)^\wedge = (-2\pi i\xi)^\alpha \hat{f} \Rightarrow$
- $\partial^\alpha f \in L^1, \forall |\alpha| \le k \sim |\xi|^k \hat{f} \in C_0.$

Define. For $m \in \mathbb{Z}_+$, $H_m = \{f \in L^2 : \partial^\alpha f \in L^2, \forall |\alpha| \le m\}$ w/ inner product

$$(f,g) \to \sum_{|\alpha| \le m} \int \partial^\alpha f \, \overline{\partial^\alpha g} \, dx$$

$\partial^\alpha f$ in sense of distributions.

One can show that H_k coincides w/ $\{f \in L^2 : \int (1+|\xi|^2)^{k/2} |\hat{f}(\xi)|^2 d\xi < \infty\}$ and that the former inner product is equivalent to

$$(f,g)_k := \int (1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi.$$

More generally, we introduce for $s \in \mathbb{R}$ the (L^2-) Sobolev space H_s as $\{F \in \mathcal{S}' : \Lambda_s F \in L^2\}$, where $\Lambda_s : \mathcal{S}' \to \mathcal{S}'$ is the isomorphism

$$\Lambda_s F = \left[(1+|\xi|^2)^{s/2} \hat{F}\right]^{\vee}.$$

H_s is a <u>Hilbert space</u> w/ inner prod.

$$(F,G)_s = \int \Lambda_s F \, \overline{\Lambda_s G} \, dx = \int (1+|\xi|^2)^s \hat{F} \hat{G} \, d\xi.$$

Prop 1. ∂^α is a bdd linear operator $H_s \to H_{s-|\alpha|}$.

Pf. $(\partial^\alpha F)^\wedge = (-2\pi i \xi)^\alpha \hat{F}$. Thus,

$(1+|\xi|^2)^{s-|\alpha|} |(\partial^\alpha F)^\wedge|^2 = (2\pi)^{2|\alpha|}$.

$|\xi|^{2|\alpha|} (1+|\xi|^2)^{s-|\alpha|} |\hat{F}|^2 \le (2\pi)^{2|\alpha|}$

$(1+|\xi|^2)^s |\hat{F}|^2$. We conclude

$$\|\partial^\alpha F\|_{H_{s-|\alpha|}} \le (2\pi)^{|\alpha|} \|F\|_{H_s} .$$ ☐

Rem. "Converse" is of course not true. If we take any $F \in \mathcal{S}'(\mathbb{R})$ and consider $\tilde{F} \in \mathcal{S}'(\mathbb{R}^2)$ given by

$\langle \tilde{F}, \varphi \rangle = \int \langle F, \varphi(\cdot, y) \rangle \, dy$ then

$$\langle \partial_y \tilde{F}, \varphi \rangle = -\langle \tilde{F}, \partial_y \varphi \rangle = -\int \langle F, \partial_y \varphi(\cdot, y) \rangle \, dy$$

$$= -\langle F, \underbrace{\int \partial_y \varphi(\cdot, y)\, dy}_{=0 \text{ by IBP.}} \rangle = 0.$$

Thus, $\partial_y \tilde{F} = 0$ but clearly $\tilde{F}$ need not even be in $H_0 = L^2$.

However, we have

<u>Sobolev Embedding Thm</u>

If $s > k + n/2$, then $H_s \subseteq \mathcal{C}_0^k$ and inclusion is continuous.

<u>Rem.</u> $s \geq 0 \Rightarrow H_s \subseteq H_0 = L^2$. The inclusion $H_s \subseteq \mathcal{C}_0^k$ means the L^2-fcns can be modified on a set of meas. 0 to be a $\mathcal{C}_0^k$ fcn.

Pf of SE Thm. We first note that if $f \in H_s$ then for $|\alpha| \le k$

$$\int |(\partial^\alpha f)^\wedge| \, d\xi = (2\pi)^{|\alpha|} \int |\xi|^{|\alpha|} |\hat{f}| \, d\xi$$

$$\le (2\pi)^k \int (1+|\xi|^2)^{\frac{k}{2}} |\hat{f}| \, d\xi$$

$$= (2\pi)^k \int (1+|\xi|^2)^{\frac{k-s}{2}} (1+|\xi|^2)^{\frac{s}{2}} |\hat{f}| \, d\xi$$

$$\le (2\pi)^k \left[\int (1+|\xi|^2)^{k-s} d\xi \right]^{1/2} \|f\|_{H_s}$$

$$k-s < -\frac{n}{2} \Rightarrow (1+|\xi|^2)^{\frac{k-s}{2}} \in L^2 \Rightarrow$$

$$\int |(\partial^\alpha f)^\wedge| \, d\xi \le (2\pi)^k \|(1+|\cdot|^2)^{\frac{k-s}{2}}\|_{L^2} \|f\|_{H_s}.$$

By Thm 8.22 (d) + Riemann-Lebesgue, it follows that $(\hat{f})^{\vee} \in C_0^k$. By Fourier inversion theorem, we conclude that

$\exists f_0 \in C_0^k$ s.t. $f = f_0$ a.e.. Moreover,

$$\|\partial^\alpha f_0\|_u \le \|(\partial^\alpha f)^\wedge\|_{L^1} \le C_{k,s} \|f\|_{H_s}$$

$\Rightarrow$ inclusion $H_s \hookrightarrow C_0^k$ is bdd. ∎

$$\boxed{\begin{array}{l}\text{An important } \underline{\text{corollary}}: \\ f \in H_s,\ \forall s \quad \Longrightarrow \quad f \in C_0^\infty.\end{array}}$$

<u>Partial differential operators (PDO)</u>.

We write $D_j = \frac{1}{2\pi i} \frac{\partial}{\partial x_j}$ and

$D^\alpha = \frac{1}{(2\pi i)^{|\alpha|}} \partial^\alpha$ so that

$$(D^\alpha f)^\wedge = \xi^\alpha \hat{f}.$$

For a linear PDO w/ constant coefficients of order $m \in \mathbb{Z}_+$

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha,$$

we denote by $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$ its <u>symbol</u> and by $P_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$ its <u>principal symbol</u>.

<u>Def.</u> $P(D)$ is <u>elliptic</u> if its principal symbol $P_m(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$.

<u>Ex</u> ① The Laplace operator

$$\Delta = \sum_{j=1}^{n} \partial_j^2 = -4\pi \sum_1^n D_j^2 \Rightarrow$$

$P_2(\xi) = -4\pi|\xi|^2 \neq 0$ for $\xi \neq 0$.

② Cauchy-Riemann operator on $\mathbb{R}^2$

$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ has principal symbol $P_1(\xi) = i\pi(\xi_1 + i\xi_2) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$.

Thus, both Δ and $\bar{\partial}$ are <u>elliptic</u>.

③ Wave operator $\square = \frac{\partial^2}{\partial t^2} - \Delta$ has

$P_2(\xi, \tau) = -4\pi(\tau^2 - |\xi|^2)$

which vanishes on the light cone $\tau^2 - |\xi|^2 = 0$. $\Rightarrow$ <u>not</u> elliptic.

Prop 1 ("Global" elliptic regularity).
Let $P(D)$ be elliptic of order $m \in \mathbb{Z}_+$.
If $u \in H_s$ and $P(D)u \in H_s$, then
$u \in H_{s+m}$.

Pf. By assumption $P(D)u = f \in H_s$.
Thus, $\hat{u} = \frac{1}{P(\xi)} \hat{f}$. Since $\hat{u} \in H_s$,
we need not worry about possible
bdd zeros of $P(\xi)$ as we shall see.

Lemma 1. P elliptic of order $m \Leftrightarrow$
$|P_m(\xi)| \geq \delta |\xi|^m$.

compact

Pf. $P_m \neq 0$ on $|\xi| = 1$ and $P_m(t\xi) =$
$t^m P_m(\xi) \Rightarrow |P_m(\xi)| \geq \delta |\xi|^m$

$$\delta = \min_{|\eta|=1} |P_m(\eta)| > 0.$$

$\boxtimes$

$$\|u\|^2_{s+m} = \int (1+|\xi|^2)^{s+m} |\hat{u}|^2 \le$$

$$\int_{|\xi|\le R} \frac{(1+|\xi|^2)^{s+m} |\hat{f}|^2}{|P(\xi)|^2} d\xi +$$

$$\int_{|\xi|>R} \frac{(1+|\xi|^2)^{s+m} |\hat{f}|^2}{\delta |\xi|^{2m}} d\xi ,$$

where $R>0$ chosen such that $|P(\xi)| \ge \delta |\xi|^m$ for $|\xi|>R$. (Recall $P(\xi) = P_m(\xi) + \sum_{|\alpha|<m} a_\alpha \xi^\alpha$ and $\sum_{|\alpha|<m} a_\alpha \frac{\xi^\alpha}{|\xi|^m} \to 0$ as $|\xi| \to \infty$.). The bdd integral is just $\int_{|\xi|\le R} (1+|\xi|^2)^{s+m} |\hat{u}|^2 d\xi$

$$\le C \|u\|^2_{H^s} < \infty.$$

For the unbdd integral, note that
$$\frac{(1+|\xi|^2)^{s+m}}{|\xi|^{2m}} \le \left(1+\frac{1}{R^2}\right)^m (1+|\xi|^2)^s$$
for $|\xi|>R$. $\Rightarrow$

$$\boxed{\|u\|^2_{H_{s+m}} \le C\|u\|_{H_s} + \frac{1}{\delta}\|f\|_{H_s}}$$

Such are called <u>elliptic estimates</u>.

We are interested in using elliptic reg. together w/ Sobolev embedding thm to deduce smoothness of solutions to elliptic PDE. Smoothness is a local property, so we want a local elliptic reg. thm.

Def. $F \in \mathcal{D}'(\Omega)$, $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, is in $H_s^{loc}(\Omega)$ (local H_s) if $\forall\, U \subset\subset \Omega \subseteq \mathbb{R}^n$ open $\exists\, G \in H_s$ s.t. $F = G$ on $\mathcal{C}_c^\infty(U)$.

Prop 2. $F \in H_s^{loc}(\Omega) \Leftrightarrow \varphi F \in H_s$, $\forall \varphi \in \mathcal{C}_c^\infty(\Omega)$

Elliptic Reg. Thm. Let $P(D)$ be elliptic of order m. If $u \in \mathcal{D}'(\Omega)$ and $P(D)u \in H_s^{loc}(\Omega)$, then $u \in H_s^{loc}(\Omega)$.

Cor 2. If $P(D)u \in \mathcal{C}^\infty(\Omega) \Rightarrow u \in \mathcal{C}^\infty(\Omega)$.

Pf of local elliptic reg. thm. $P=P(D)$

We have $Pu \in H_s^{loc}(\Omega)$ and need to show $\forall \varphi \in C_c^\infty(\Omega)$, $\varphi u \in H_{s+m}$.

The idea is as follows. Since $\varphi u \in \mathcal{E}'$, $(\varphi u)^\wedge$ is C^∞ of slow growth $\Rightarrow$ $\varphi u \in H_\sigma$ for some $\sigma \in \mathbb{R}$. We first show that if $\sigma < s+m$ then $\varphi u \in H_{\sigma+1}$.

Let $[P,\varphi]f = P(\varphi f) - \varphi P f$. We note $[P,\varphi]$ is PDO w/ C_c^∞ coeff's of order $\leq m-1$. Thus,

$$P(\varphi u) = \underbrace{\varphi P u}_{\in H_s} - \underbrace{[P,\varphi]u}_{H_{\sigma-m+1}}$$

Thus, if $\sigma - m + 1 \leq s$ $(\sigma < s+m)$

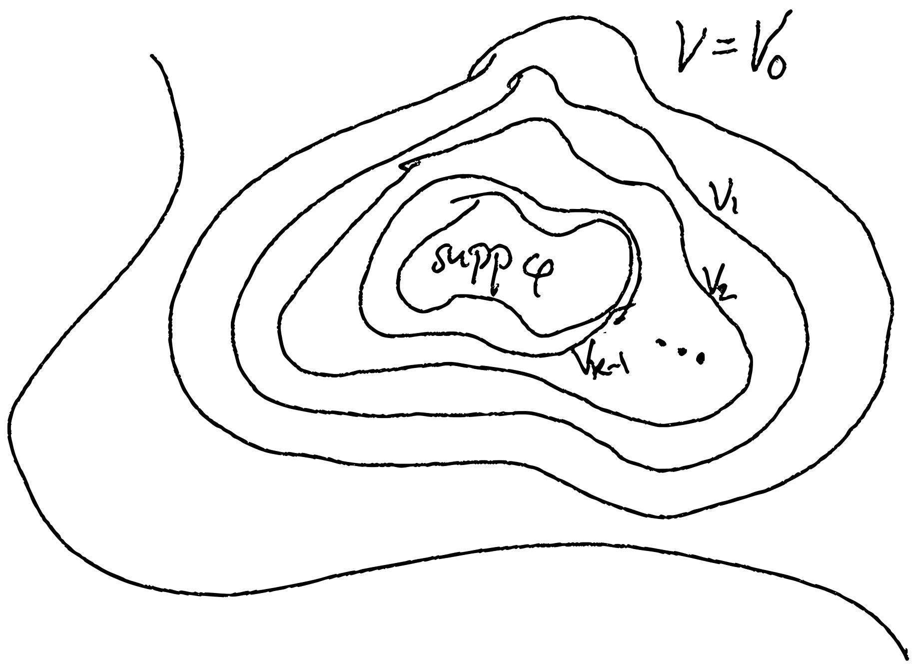
then RHS $\in H_{\sigma-m+1}$ so

Global Elliptic regularity thm $\Rightarrow$

$\varphi u \in H_{\sigma-m+1+m} = H_{\sigma+1}$.

Since $\varphi \in C_c^\infty(\Omega)$ arbitrary, we can devise a finite inductive argument that bumps up regularity until we reach $\sigma = s+m$ where process stops.

<u>Inductive argument</u>. Fix $\varphi \in C_c^\infty(\Omega)$, pick $\psi \in C_c^\infty(\Omega)$ s.t. $\operatorname{supp}\varphi \subset\subset V \subset\subset \Omega$ and $\psi = 1$ on $\overline{V}$. Then, $\psi u \in H_\sigma$ for some fixed σ (WLOG $\sigma < s+m$) and set $\alpha = s+m-\sigma \in \mathbb{Z}_+$



Choose $\operatorname{supp}\varphi \subset\subset V_{\kappa-1} \subset\subset \dots \subset\subset V_1 \subset\subset V_0 = V$ and $\psi_j \in \mathcal{C}_c^\infty(V_{j-1})$, $\psi_j = 1$ on $\overline{V_j}$ for $1 \le j \le \kappa - 1$, and set $\psi_0 = \psi$ and $\psi_\kappa = \varphi$. Repeating the "bootstrap" above on $\psi_\ell \cdots \psi_1 \psi_0 u = \psi_\ell u$ for $\ell = 0, \dots, \kappa$ gives $\psi_\ell u \in H_{\sigma+\ell}$ and $\varphi u = \psi_\kappa u \in H_{\sigma+\kappa} = H_{s+m}$. ∎